Common Core Algebra 2

Chapter 2: Quadratic Functions

2.1 – Transformations of Quadratic Functions

Essential Question:

How do constants a, h and k affect the graph of the quadratic function $g(x) = a(x - h)^2 + k$?

What You Will Learn

- Describe transformations of quadratic functions.
- Write transformations of quadratic functions.

A **quadratic** is a function that *one* of the ways can be written in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The U-shaped graph of a quadratic function is called a **parabola.**

Describing Transformations of Quadratic Functions

1. Describe the transformation of $f(x) = x^2$ represented by $g(x) = (x + 4)^2 - 1$.

Transformation(s):

Vertex of f:

Vertex of g:

Notice that the function is of the form $g(x) = a(x - h)^2 + k$. Rewrite $g(x)$ to identify h and k.

2. Describe the following transformations of $f(x) = x^2$ represented by g. Then identify the vertex.

a) $g(x) = (x - 3)^2$

b) $g(x) = (x - 2)^2 - 2$

c) $g(x) = (x + 5)^2 + 1$

3. Describe the following transformations of $f(x) = x^2$ represented by g. Then identify the vertex.

a) $g(x) = -\frac{1}{2}x^2$

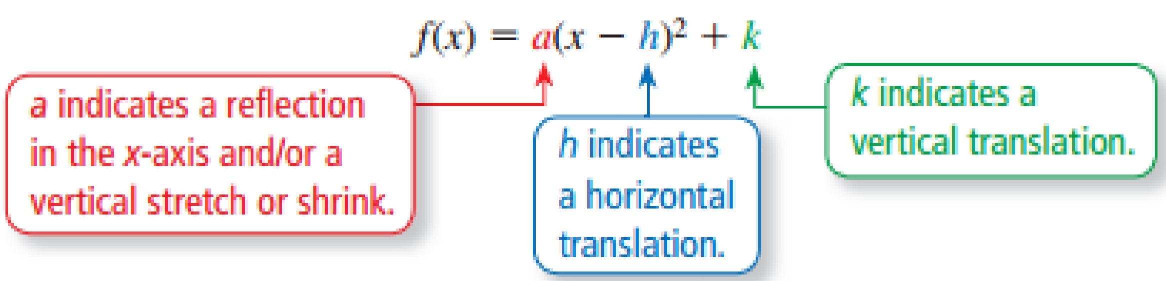
b) $g(x) = (2x)^2 + 1$

c) $g(x) = \left(\frac{1}{3}x\right)^2$

d) $g(x) = 3(x-1)^2$

e) $g(x) = -(x+3)^2 + 2$

The lowest point on a parabola that opens up or the highest point on a parabola that opens down is the **vertex**. Writing a quadratic function in the form $g(x) = a(x-h)^2 + k$ where $a \neq 0$ is known as **vertex form**.



Writing a Transformed Equation

4. Let the graph of g be a vertical stretch by a factor of 2 and a reflection in the $x - axis$, followed by a translation 3 units down of the graph $f(x) = x^2$. Write a rule for g.

5. Let the graph of g be a translation 3 units right and 2 units up, followed by a reflection in the $y - axis$ of the graph $f(x) = x^2 - 5x$. Write a rule for g.

6. Let the graph of g be a vertical shrink by a factor of $\frac{1}{2}$ followed by a translation 2 units up of the graph $f(x) = x^2$. Write the rule for g and identify the vertex.

7. Let the graph of g be a translation 4 units left followed by a horizontal shrink by a factor of $\frac{1}{3}$ of the graph $f(x) = x^2 + x$. Write a rule for g and identify the vertex.

Modeling with Mathematics

8. The height h (in feet) of water spraying from a fire hose can be modeled by $h(x) = -0.03x^2 + x + 25$, where x is the horizontal distance (in feet) from the fire truck. The crew raises the ladder so that the water hits the ground 10 feet farther from the fire truck. Write a function that models the new path of the water.

To solve this problem, let's first look at the original function $h(x)$:

Sketch of the graph

At what distance does the water originally hit the ground? ________________________

If we want the water to hit the ground 10 feet farther, what $x -$ value should we look at? _________________

New Equation:

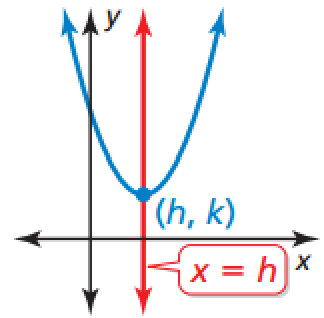
2.2 – Characteristics of Quadratic Functions

Essential Question – What type of symmetry does the graph of $f(x) = a(x-h)^2 + k$ have and how can you describe this symmetry?

What Will You Learn

- Explore properties of parabolas.
- Find maximum and minimum values of quadratic functions.
- Graph quadratic functions using x-intercepts.
- Solve real-life problems.

Exploring Properties of Parabolas

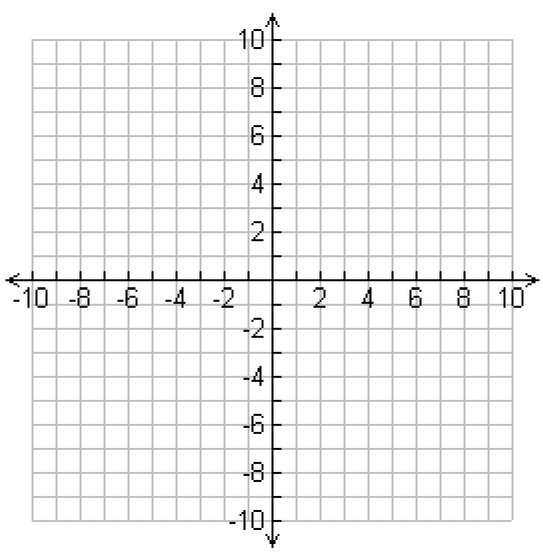


An **axis of symmetry** is a line that divides a parabola into mirror images and passes through the vertex.

Because the vertex of $f(x) = a(x-h)^2 + k$ is ____________________,

the axis of symmetry is $x =$ ________.

1. Graph $f(x) = -2(x+3)^2 + 4$. Label the vertex and axis of symmetry.



Quadratic equations can also be written in **standard form**, $f(x) = ax^2 + bx + c$ where $a \neq 0$. We can derive standard form by expanding vertex form.

$$f(x) = a(x-h)^2 + k$$

$$f(x) = ax^2 + (-2ah)x + (ah^2 + k)$$

$$f(x) = ax^2 + bx + c$$

This allows us to make the following observations.

$\boldsymbol{a = a}$: So a, has the same meaning in vertex form as it does in standard form.

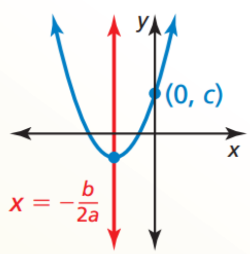
$\boldsymbol{b = -2ah}$: Solve for h to obtain ____________. So the axis of symmetry is $x =$ ______.

$\boldsymbol{c = ah^2 + k}$: In vertex form $f(x) = a(x-h)^2 + k$, notice that $f(0) =$ ______________.
So c is the y-intercept.

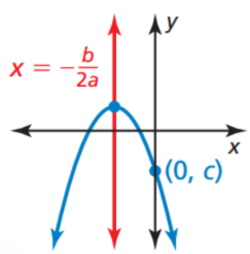
Core Concept

Properties of the Graph of $f(x) = ax^2 + bx + c$

$y = ax^2 + bx + c, a > 0$

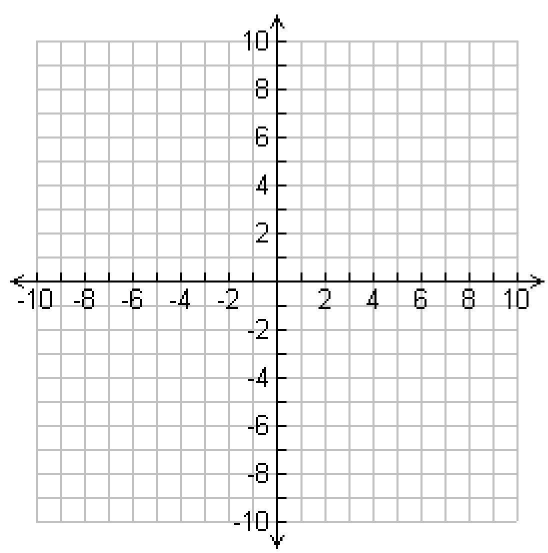


$y = ax^2 + bx + c, a < 0$



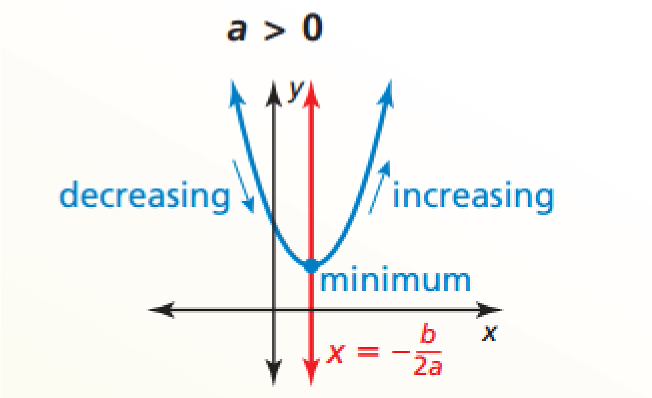
- The parabola opens up when $a > 0$ and opens down when $a < 0$.
- The graph is narrower than the graph of $f(x) = x^2$ when $|a| > 1$ and wider when $|a| < 1$.
- The axis of symmetry is $x = -\frac{b}{2a}$ and the vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
- The y-intercept is c. So, the point $(0, c)$ is on the parabola.

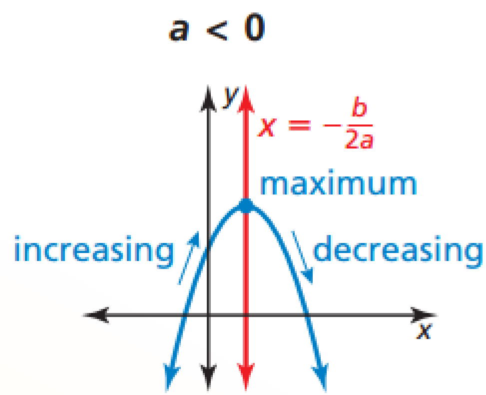
2. Graph $f(x) = 3x^2 - 6x + 1$. Label the vertex and axis of symmetry.



FINDING MAXIMUM AND MINIMUM VALUES

For the quadratic function $f(x) = ax^2 + bx + c$, the $y - coodinate$ of the vertex is the *minimum value* of the function when $a > 0$ and the *maximum value* when $a < 0$.





Minimum Value: ____________________

Domain: ____________________________

Range: _____________________________

Decreasing to the _________ of $x = -\frac{b}{2a}$

Increasing to the __________ of $x = -\frac{b}{2a}$

Maximum Value: ___________________

Domain: ___________________________

Range: ____________________________

Increasing to the _________ of $x = -\frac{b}{2a}$

Decreasing to the _________ of $x = -\frac{b}{2a}$

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3. For each equation, find the vertex and the equation of the axis of symmetry. Then state whether the vertex is a maximum or minimum.

a) $f(x) = -3(x+1)^2$

b) $g(x) = 2(x-2)^2 + 5$

c) $h(x) = x^2 + 2x - 1$

d) $p(x) = -2x^2 - 8x + 1$

4. Find the minimum value or maximum value of the following functions. Describe the domain and range of each function, and where each function is increasing and decreasing.

(a) $f(x) = 4x^2 + 16x - 3$

(b) $h(x) = -x^2 + 5x + 9$

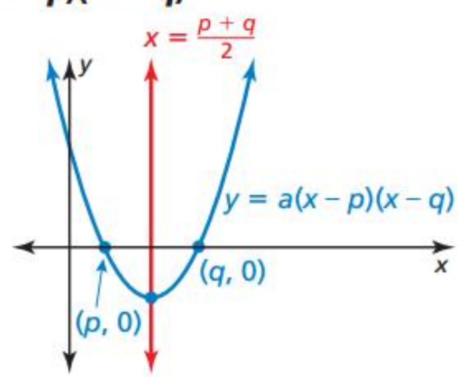
GRAPHING QUADRATIC FUNCTIONS USING X-INTERCEPTS

When the graph of a quadratic function has at least one $x - intercept$, the function can be written in **intercept form**, $f(x) = a(x-p)(x-q)$, where $a \neq 0$.

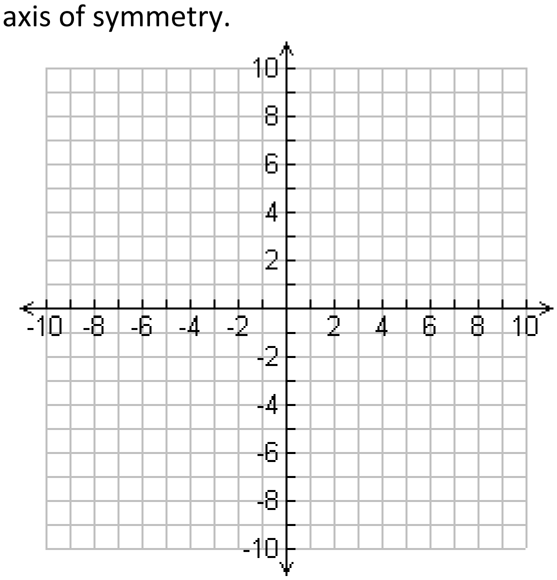
Core Concept

Properties of the Graph of $f(x) = a(x - p)(x - q)$

- Because $f(p) = 0$ and $f(q) = 0$, p and q are the x-intercepts of the graph of the function.
- The axis of symmetry is halfway between $(p, 0)$ and $(q, 0)$. So, the axis of symmetry is $x = \dfrac{p+q}{2}$.
- The parabola opens up when $a > 0$ and opens down when $a < 0$.

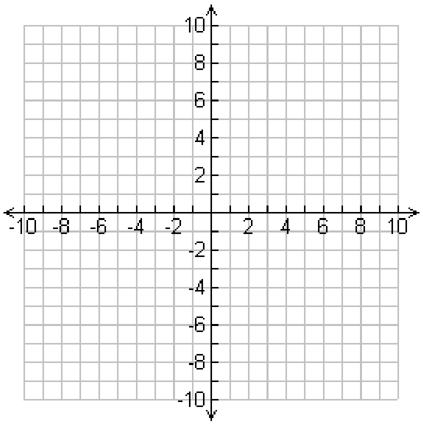


5. Graph $f(x) = -2(x+3)(x-1)$. Label the x-intercepts, vertex, and axis of symmetry.

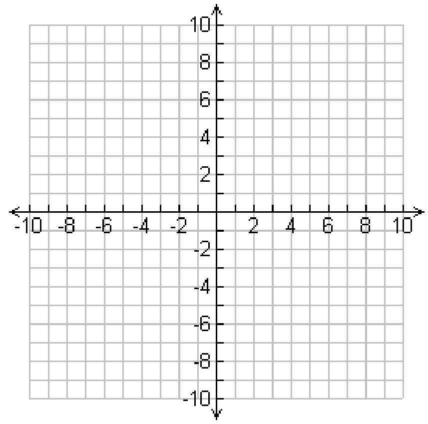


6. Graph each of the following functions. Label the x-intercepts, vertex, and axis of symmetry.

(a) $f(x) = -(x+1)(x+5)$

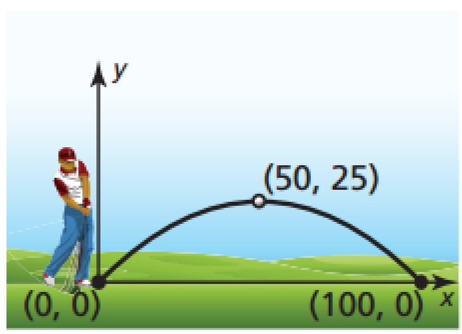


(b) $g(x) = \frac{1}{4}(x-6)(x-2)$



Modeling with Mathematics

7. The parabola shows the path of your first golf shot, where x is the horizontal distance (in yards) and y is the corresponding height (in yards).



The path of your second shot can be modeled by the function: $f(x) = -0.02x(x-80)$. Which shot travels farther before hitting the ground? Which travels higher?

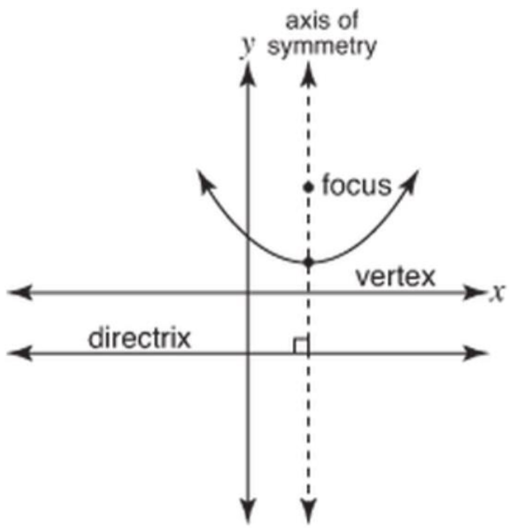
2.3 – Focus of a Parabola

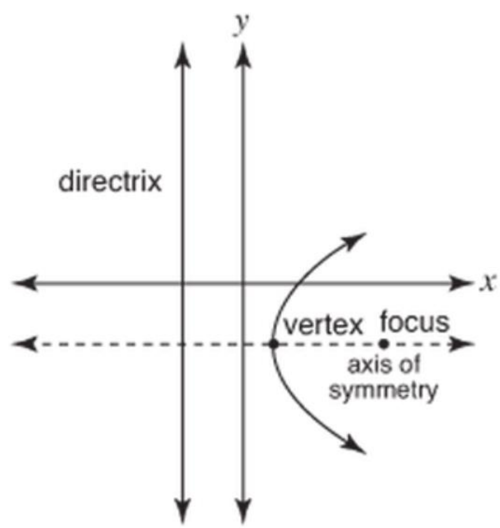
Essential Question: What is the focus of a parabola?

What You Will Learn:

- Explore the focus and the directrix of a parabola.
- Write equations of parabolas.
- Solve real-life problems.

EXPLORING THE FOCUS AND DIRECTRIX





EXPLORATION:

Materials: Patty Paper, Graph Paper, and Ruler

Step 1: Draw a dark, black line on the bottom of your patty paper and label the line d (for directrix).

Step 2: About 3 centimeters above the directrix mark a black dot and label it F (for focus).

Step 3: Draw a few points on line d.

Step 4: At each point, neatly fold the paper so that the point F lies directly on the line.

Teacher Demonstration: http://www.youtube.com/watch?v=wtk5q8wGAe0

Questions:

1. What is the name of the shape that is formed by all the folds in this activity?

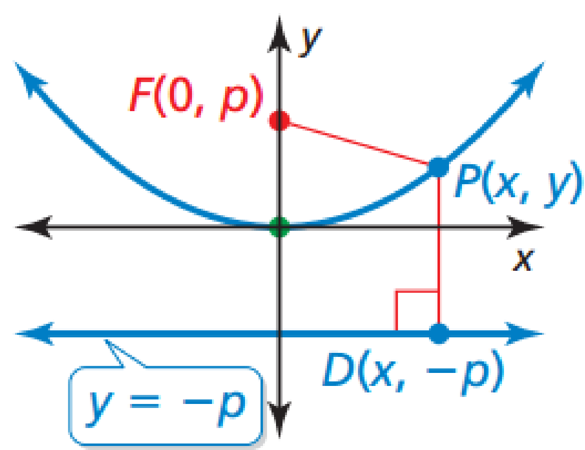
2. Fill in the blank: The vertex is ________________ between the _____________________ and the _________________________.

3. The directrix is _______________________________ to a parabolas axis of symmetry.

A **parabola** is the set of points (locus of points) that are equidistant from a given point and a given line in a plane. The given point is called the **focus**, and the line is called the **directrix**.

The midpoint on the perpendicular segment from the focus to the directrix is call the **vertex of the parabola**. The line that passes through the vertex and focus is called the **axis of symmetry**.

We can derive the equation of a parabola that opens up or down with vertex $(0,0)$, focus $(0,p)$, and directrix $y = -p$ using the distance formula.



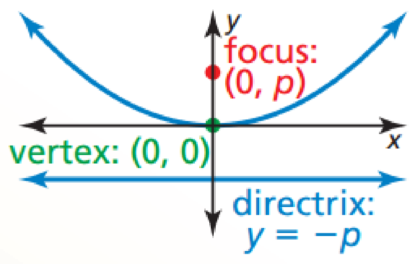
Standard Equations of a Parabola with Vertex at the Origin

Vertical axis of symmetry ($x = 0$)

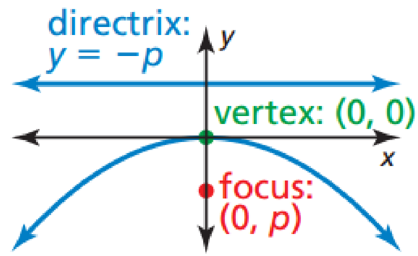
Equation: $y = \frac{1}{4p}x^2$

Focus: $(0, p)$

Directrix: $y = -p$



$p > 0$



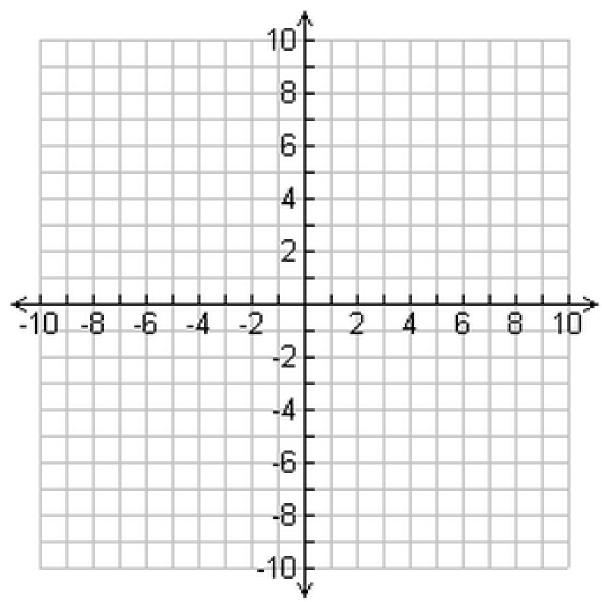
$p < 0$

WRITING EQUATIONS OF PARABOLAS

1. Write an equation of the parabola with the given characteristics.

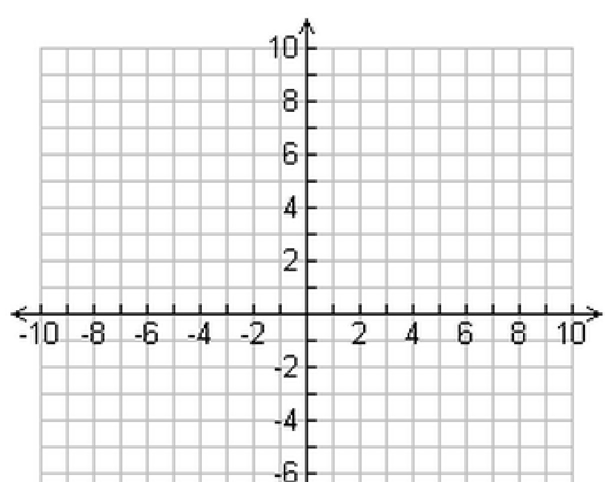
Vertex: $(0, 0)$

Directrix: $y = -6$



2. Directrix: $y = 7$

   Focus: $(0, -7)$

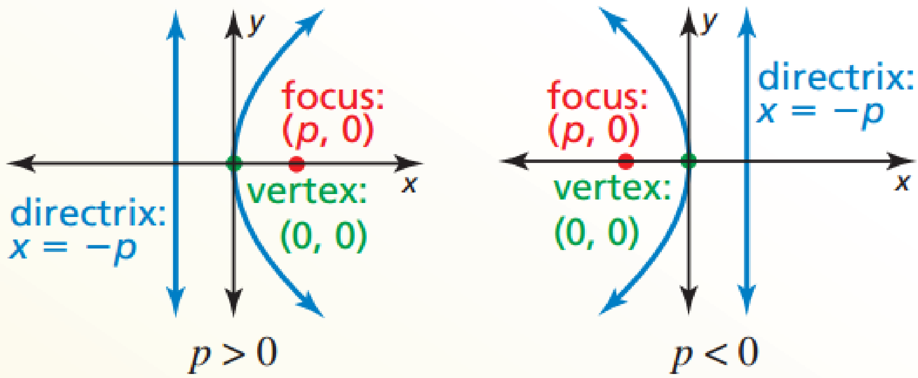


Horizontal axis of symmetry ($y = 0$)

Equation: $x = \frac{1}{4p}y^2$

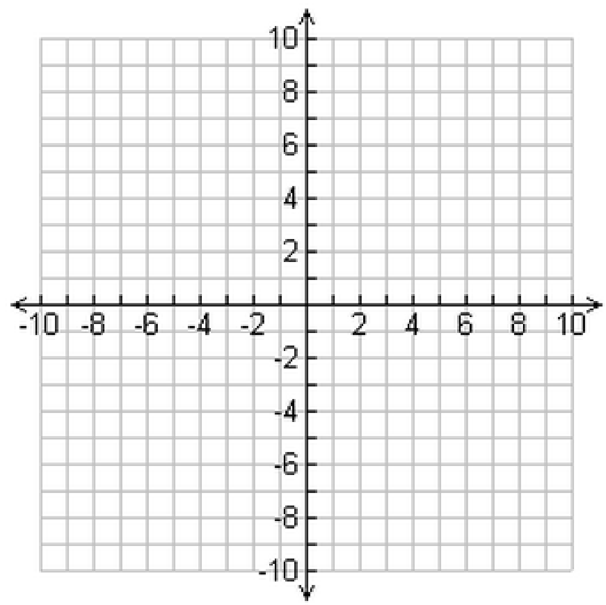
Focus: $(p, 0)$

Directrix: $x = -p$



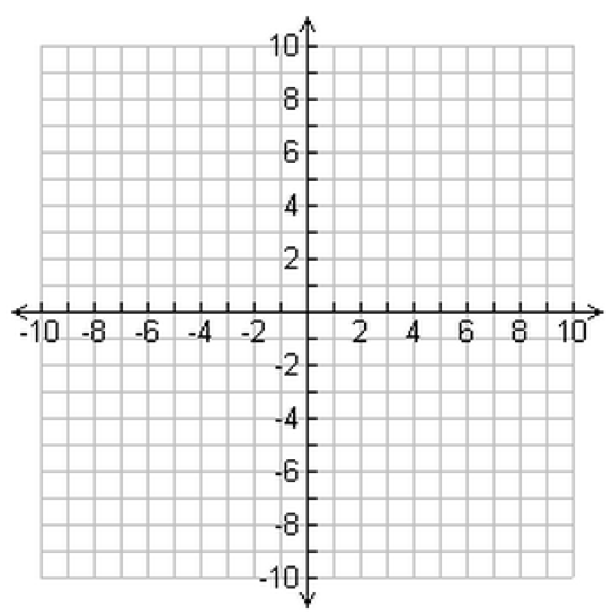
3. Directrix: $x = -3$

   Focus: $(3, 0)$



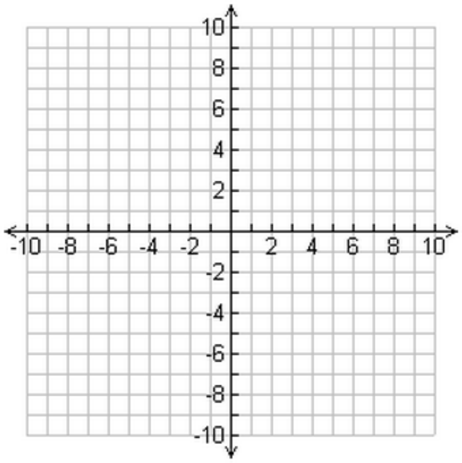
4. Vertex: $(0, 0)$

   Focus: $(-5, 0)$

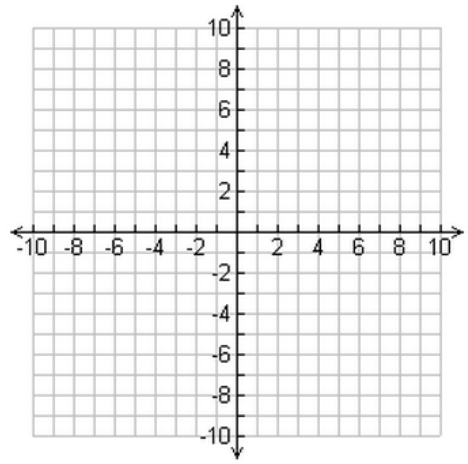


5. Graph each of the following equations. Identify the focus, directrix, vertex, and equation of the axis of symmetry of each of the following parabolas.

(a) $y = \frac{1}{10}x^2$

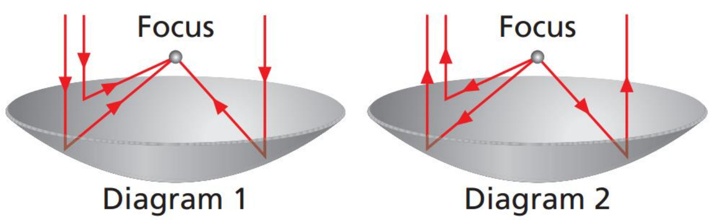


(b) $y^2 = 16x$

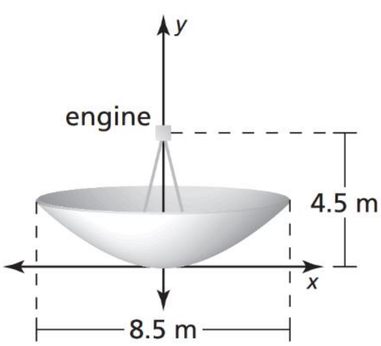


SOLVING REAL-LIFE PROBLEMS

Parabolic reflectors have cross sections that are parabolas. Incoming sound, light, or other energy that arrives at a parabolic reflector parallel to the axis of symmetry is directed to the focus (Diagram 1). Similarly, energy that is emitted from the focus of a parabolic reflector and then strikes the reflector is directed parallel to the axis of symmetry (Diagram 2).



6. An electricity generating dish uses a parabolic reflector to concentrate sunlight onto a high-frequency engine located at the focus of the reflector. The sunlight heats helium to 650°C to power the engine. Write an equation that represents the cross section of the dish shown with its vertex at (0, 0). What is the depth of the dish?



The vertex of a parabola is not always at the origin. As in previous transformations, adding a value to the input or output of a function translates the graph.

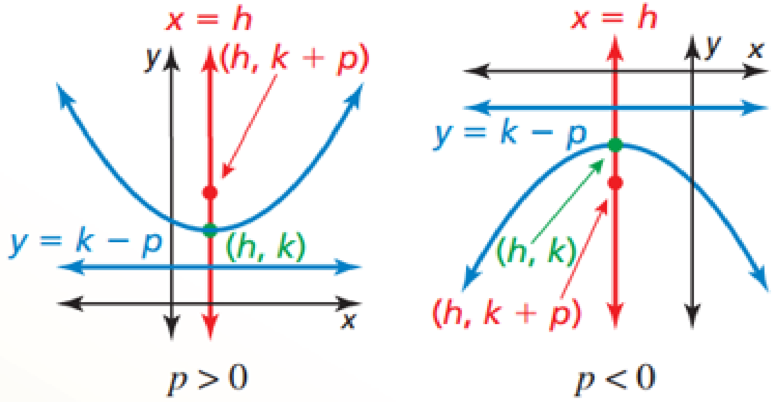
Standard Equations of a Parabola with Vertex at (h, k)

Vertical axis of symmetry ($x = h$)

Equation: $y = \dfrac{1}{4p}(x - h)^2 + k$

Focus: $(h, k + p)$

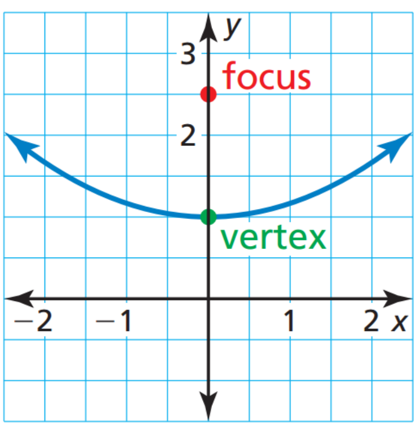
Directrix: $y = k - p$



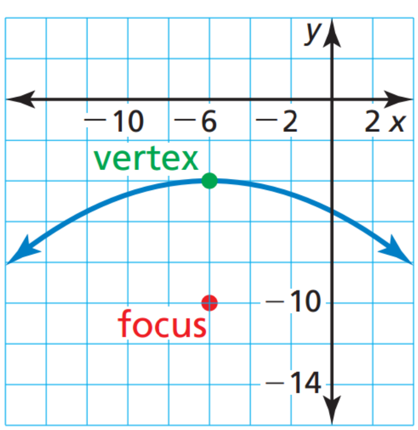
WRITING EQUATIONS OF PARABOLA WITH VERTEX (h, k)

7. Write the equation of each of the parabolas shown below.

(a)



(b)

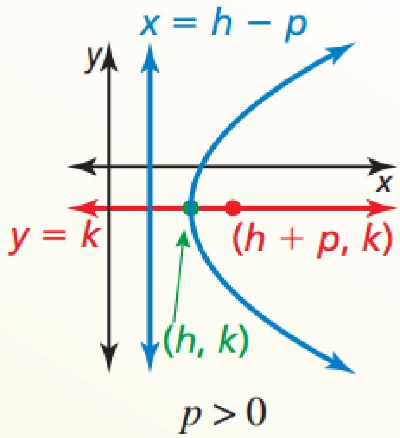


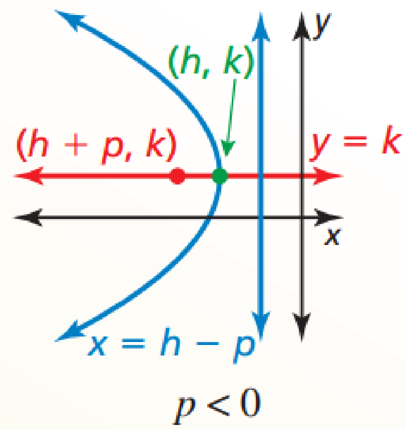
Horizontal axis of symmetry ($y = k$)

Equation: $x = \frac{1}{4p}(y - k)^2 + h$

Focus: $(h + p, k)$

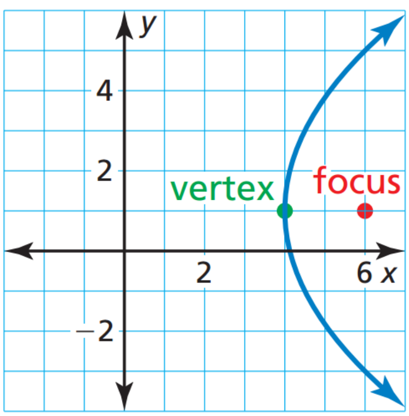
Directrix: $x = h - p$



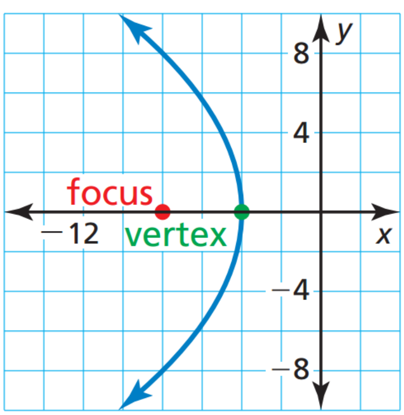


8. Write the equation of each of the parabolas shown below.

(a)



(b)



9. Identify the vertex, focus, directrix, and axis of symmetry of the following parabolas.

(a) $y = \frac{1}{8}(x-3)^2 + 2$

(b) $y = -\frac{1}{4}(x+2)^2 + 1$

(c) $x = \frac{1}{16}(y-3)^2 + 1$

(d) $x = -3(y+4)^2 + 2$

<u>Check this out</u>:

http://www.mathwarehouse.com/quadratic/parabola/focus-and-directrix-of-parabola.php

2.4 – Modeling with Quadratic Functions

<u>Essential Question</u>: How can you use a quadratic function to model a real-life situtation?

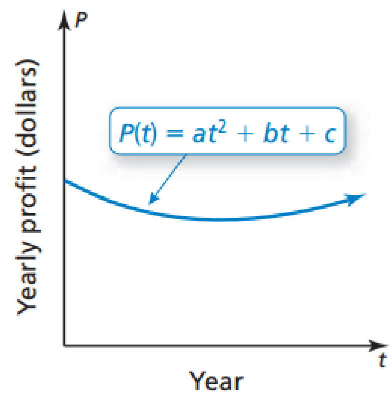
What You Will Learn

- ➢ Write equations of quadratic functions using vertices, points, and x-intercepts.
- ➢ Write quadratic equations to model data sets.

EXPLORATION 1 - Modeling with a Quadratic Function

The graph shows a quadratic function of the form:

$$P(t) = at^2 + bt + c$$



a) Is the value of a positive, negative, or zero?

b) What characteristic of the quadratic function will help us find the year t when the company made the least profit?

c) Write the equation you would need to find the value of t in part b.

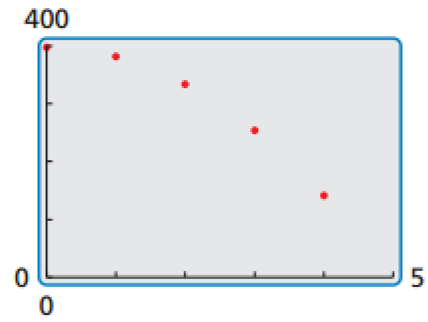
d) The company made same yearly profits in 2004 and 2012. Estimate the year in which the company made the least profit.

e) Assume that the model is still valid today. Are the yearly profits currently increasing, decreasing, or constant?

EXPLORATION 2 – Modeling with a Graphing Calculator

The table shows the heights h (in feet) of a wrench t seconds after it has been dropped from a building under construction.

Time, t	0	1	2	3	4
Height, h	400	384	336	256	144



a) Using a graphing calculator to create a scatter plot of the data.

b) What type of regression would best fit this data?

c) Find the equation of the regression that best represents this data?

d) Graph this equation on the same screen as the scatter plot to verify that it fits the data.

e) When does the wrench hit the ground? (Think about what part of the quadratic function this represents.)

Writing Quadratic Equations

Core Concept

Writing Quadratic Equations

Given a point and the vertex (h, k) — Use vertex form:

$$y = a(x - h)^2 + k$$

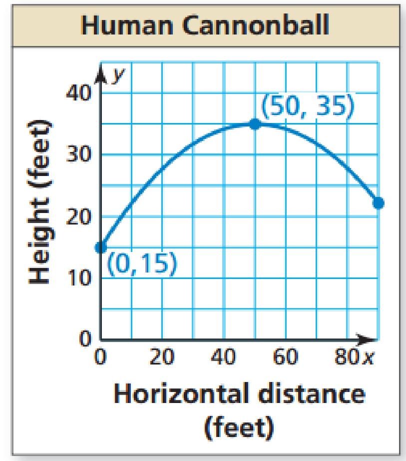
Given a point and x-intercepts p and q — Use intercept form:

$$y = a(x - p)(x - q)$$

Given three points — Write and solve a system of three equations in three variables.

Example 1: The graph shows the parabolic path of a performer who is shot out of a cannon, where y is the height (in feet) and x is the horizontal distance traveled (in feet).

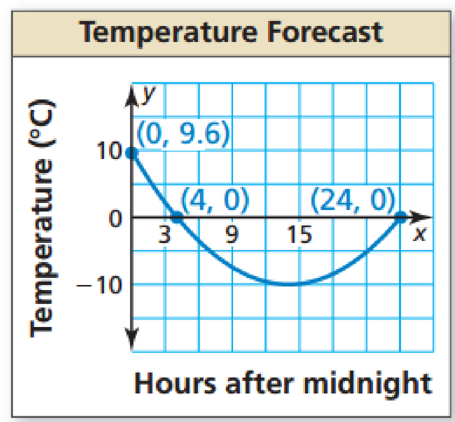
a) Write an equation of the parabola.



b) If the performer lands in a net 90 feet from the cannon, what is the height of the net?

Example 2: A meteorologist creates a parabola to predict the temperature tomorrow, where x is the number of hours after midnight and y is the temperature (in degrees Celsius).

a) Write a function f that models the temperature over time.



b) What is the coldest temperature?

c) What is the average rate of change in temperature over the interval in which the temperature is decreasing? increasing? Compare the average rates of change.

WRITING EQUATIONS TO MODEL DATA

When data have equally –spaced inputs, you can analyze patters in the differences of the outputs to determine what type of function that can be used to model the data.

Linear data have constant *first differences.*

x	-3	-2	-1	0	1	2	3
y	-5	-3	-1	1	3	5	7

Quadratic data have constant *second differences*.

Equally-spaced x-values

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9

first differences:

second differences:

Example 3: NASA can create a weightless environment by flying a plane in parabolic paths. The table shows heights h (in feet) of a plane t seconds after starting the flight path.

Time, t	Height, h
10	26,900
15	29,025
20	30,600
25	31,625
30	32,100
35	32,025
40	31,400

a) Using the table show that a quadratic function is an appropriate model to best represent this data.

b) Write a quadratic equation in the form $h(t) = at^2 + bt + c$ that models the data. Use any three points (t, h) from the table.

c) After about 20.8 seconds, passengers begin to experience a weightless environment. Use your equation to approximate the height at which this occurs.

Real-life data that show a quadratic relationship usually do not have constant second differences because the data are not exactly quadratic. Relationships that are *approximately* quadratic have second differences that are relatively "close" in value. Your graphing calculator has a quadratic regression feature that you can use to find a quadratic function that best models a set of data.

Example 4: The table shows fuel efficiencies of a vehicle at different speeds. Write a function that models the data. Use the model to approximate the optimal driving speed.

Miles per hour, *x*	**Miles per gallon, *y***
20	14.5
24	17.5
30	21.2
36	23.7
40	25.2
45	25.8
50	25.8
56	25.1
60	24.0
70	19.5

Practice Problems:

5. Write an equation of the parabola that passes through the points $(-1, 4)$, $(0, 1)$, and $(2, 7)$.

6. The table shows the estimated profits y (in dollars) for a concert when the charge is x dollars per ticket. Write and evaluate a function to determine what the charge per ticket should be to maximize the profit.

Ticket price, *x*	2	5	8	11	14	17
Profit, *y*	2600	6500	8600	8900	7400	4100

7. The table shows the results of an experiment testing the maximum weights y (in tons) supported by ice x inches thick. Write a function that models the data. How much weight can be supported by ice that is 22 inches thick?

Ice thickness, *x*	12	14	15	18	20	24	27
Maximum weight, *y*	3.4	7.6	10.0	18.3	25.0	40.6	54.3